Exponential Distribution

A Continuous RV X is said to follow an exponential distribution with parameter $\alpha > 0$, if its PDF is given by
$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Derive MGF, Mean and variance of Exponential distribution

The PDF of Exponential distribution is
$$f(x) = \alpha e^{-\alpha x}, \quad x > 0.$$

$$MGF = M_x(t) = E[e^{tx}]$$

$$= \int e^{tx} f(x)\, dx.$$

$$= \alpha \int_0^\infty e^{tx} e^{-\alpha x}\, dx = \alpha \int_0^\infty e^{(t-\alpha)x}\, dx.$$

$$= \alpha \int_0^\infty e^{-(\alpha - t)x}\, dx.$$

$$= \alpha \left[\frac{e^{-(\alpha-t)x}}{-(\alpha-t)} \right]_0^\infty = \alpha \left[0 - \frac{1}{-(\alpha-t)} \right]$$

$$\boxed{M_x(t) = \frac{\alpha}{\alpha - t}}$$

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$$\text{Mean} = E[x] = \left[M_x'(t)\right]_{t=0}$$

$$= \left\{\frac{d}{dt}\left[\alpha(\alpha-t)^{-1}\right]\right\}_{t=0}$$

$$= \left[-\alpha(\alpha-t)^{-2}(-1)\right]_{t=0}.$$

$$= \frac{\alpha}{\alpha^2} = \frac{1}{\alpha}.$$

$$\boxed{\text{Mean} = \frac{1}{\alpha}}$$

$$E[x^2] = \left[M_x''(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\alpha(\alpha-t)^{-2}\right)\right]_{t=0}$$

$$= \left[-2\alpha(\alpha-t)^{-3}(-1)\right]_{t=0}.$$

$$= \left[\frac{2\alpha}{(\alpha-t)^3}\right]_{t=0} = \frac{2\alpha}{\alpha^3}$$

$$E(x^2) = \frac{2}{\alpha^2}.$$

$$\therefore \text{variance} = E(x^2) - \left[E(x)\right]^2$$

$$= \frac{2}{\alpha^2} - \frac{1}{\alpha^2}$$

$$\boxed{\text{Variance} = \frac{1}{\alpha^2}}$$

State and prove Memory less Property of Exponential distribution.

If x is exponentially distributed then

$$P[x > s+t / x > s] = P[x > t] \text{ for any } s, t > 0$$

$$f(x) = \alpha e^{-\alpha x}, \quad x > 0$$

Consider $P(x > k) = \int_{k}^{\infty} f(x)\,dx = \alpha \int_{k}^{\infty} e^{-\alpha x}\,dx$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha} \right]_{k}^{\infty} = \alpha \left[0 + \frac{e^{-k\alpha}}{\alpha} \right]$$

$$P(x > k) = e^{-k\alpha} \quad \text{—— (1)}.$$

$$P[x > s+t / x > s] = \frac{P[x > s+t \cap x > s]}{P[x > s]} \quad \left[\because P(A/B) = \frac{P(A \cap B)}{P(B)} \right]$$

$$= \frac{P[x > s+t]}{P(x > s)}$$

$$= \frac{e^{-\alpha(s+t)}}{e^{-\alpha s}} \quad (\because \text{by } (1))$$

$$= \frac{e^{-\alpha s} e^{-\alpha t}}{e^{-\alpha s}} = e^{-\alpha t}$$

$$= e^{-\alpha t} = P(x > t) \quad (\text{by } ①)$$

$$\therefore \; P\left(x > s+t / x > s\right) = P(x > t).$$

Hence the proof.